MIT OCW Real Analysis

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Contents

Part I

Assignment 1

Exercise 0.3.6

> Prove:
> *a)* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
> *b)* $A \cup (B \cap C) = (A \cup B) \cap (a \cup C)$

a) In order to prove this equivalence, we have to prove the implication both ways. We use two lemmas for this.

Lemma 1.1. $A \cap (B \cup C) \implies (A \cap B) \cup (A \cap C)$

Let $x \in A \cap (B \cup C)$. By the definition of set intersection, $x \in A$ and $x \in B \cup C$. By the definition of set union, $x \in A$ and ($x \in B$ or $x \in C$). From propositional logic we know that for propositions P, Q and R the following holds: $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$. So, substituting for this particular case yields ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). Using the definition of set intersection again gets $x \in A \cap B$ or $x \in A \cap C$. Using the definition of set union again gives $x \in (A \cap B) \cup (A \cap C)$. □

Lemma 1.2. $(A \cap B) \cup (A \cap C) \implies A \cap (B \cup C)$

Let $x \in (A \cap B) \cup (A \cap C)$. By the definition of set union, $x \in (A \cap B)$ or $x \in (A \cap C)$. By the definition of set intersection, ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in B$). Using the same propositional logical equivalence as in Lemma 1.1, this gives $x \in A$ and ($x \in B$ or $x \in C$). Wrapping up, we use the definition of set union to get $x \in A$ and $x \in B \cup C$ and the definition of intersection to get $x \in A \cap (B \cup C)$. □

Using Lemma 1.1 and 1.2, we get the desired equivalence of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

b) This proof is so similar to a) that it feels like a waste of time and will therefore be left to the reader.

Exercise 0.3.11

> Prove by induction that $n < 2^n$ for all $n \in \mathbb{N}$.

For this proof we will use induction. For this, we have to prove the base case, i.e. $n = 1$, and the inductive step, $n < 2^n \implies n + 1 < 2^{n+1}$.

First, let's prove the base case. When $n = 1$, we get $1 < 2^1$, which is certainly true.

Then, for the inductive step. We assume that the proposition holds for any $m \in \mathbb{N}$. So, $m < 2^m$. Multiplying both sides with 2 gives $2m < 2^{m+1}$. Since $m \geq 1$, $m + 1 < 2m$, and thus $m + 1 < 2m < 2^{m+1}$.

Since both the base case and inductive step hold, we can close the induction, proving the proposition. □

Exercise 0.3.12

> Show that for a finite set A of cardinality n, the cardinality of $\mathcal{P}(A)$ is 2^n.

The power set of a set A, $\mathcal{P}(A)$, is defined as the set of all possible subsets of A. This is very similar to an inclusion/exclusion problem. It is built up by all the possible combinations of the different elements being either inside a certain subset or not. For all possible subsets of A, we have that for every element $x \in A$ there are 2 possibilities, either x is in the subset or it isn't. This means that for every additional element, the number of subsets increases by a factor of 2, with a minimum of 1, in case of $A = \emptyset$. We will prove this formally now, using induction.

For this, the base case is a set of 1 element (but the theorem also holds for the empty set, where $n = 0$). Let us assume that $A := \{\pi\}$. Then the cardinality of $\mathcal{P}(A)$ is 2^1, with $\mathcal{P}(A) = \{\emptyset, \{\pi\}\}$.

For the inductive step, we assume that for any set B of cardinality m, the cardinality of the power set of B is 2^m. Then, we will add an element $x \notin B$ to B to increase its cardinality by 1, to $m + 1$, creating a new set C. Note that all the possible subsets of B are still viable subsets of C, since $B \subset C$. In order to create the new subsets, we can simply keep all the subsets of B, duplicate them and take the union with the new element x, so now we also have all combinations of the old sets with possibly x being in them. Since this doubles the number of subsets, the cardinality of $\mathcal{P}(C)$ is 2^{m+1}.

Both the base case and the inductive step hold, which closes the induction and proves the proposition. □

Exercise 0.3.15

> Prove that $n^3 + 5n$ is divisible by 6 for all $n \in \mathbb{N}$.

In order to prove this proposition, we will use induction. To do this, we need to prove the following lemma, of which we will see the usefulness later:

Lemma 4.1. $3n^2 + 3n + 6$ *is divisible by 6 for all* $n \in \mathbb{N}$.

This lemma we will also prove by induction. For this, we prove the base case and the inductive step. First, for the base case we have $n = 1$, yielding $3 \cdot 1^2 + 3 \cdot 1 + 6 = 12$, which is divisibly by 6.

Then, for the inductive step we assume that the lemma holds for a certain $m \in \mathbb{N}$. So, $3m^2 + 3m + 6$ is divisible by 6. Substituting m with $m + 1$ gives $3(m + 1)^2 + 3(m + 1) + 6$, which can be expanded to $3m^2 + 9m + 12$. Rewriting this with our assumption in mind gives the following: $(3m^2 + 3m + 6) + (6m + 6)$. We know from our assumption that the first part is divisible by 6, and since $m \in \mathbb{N}$, $6m + 6$ is also divisible by 6, and so the whole expression is as well. □

Now for the original proposition. We will prove this by induction. First we prove the base case, where $n = 1$. Then, $1^3 + 5 \cdot 1 = 6$, which is definitely divisible by 6.

For the inductive step, we assume that the proposition holds for a certain $m \in \mathbb{N}$. So, $m^3 + 5m$ is divisible by 6. When we increase m by 1, we get: $(m+1)^3 + 5(m+1)$. Expanded, this is the same as $m^3 + 3m^2 + 8m + 6$. When we rearrange the terms we can get the following expression: $(m^3 + 5m) + (3m^2 + 3m + 6)$. From Lemma 4.1, we know that the latter part is divisible by 6. The prior part is divisible by 6 because of the assumption of the inductive step. So together, this expression is also divisible by 6. □

Exercise 0.3.19

Give an example of a countably infinite collection of finite sets $A_1, A_2, ...$, whose union is not a finite set.

The easiest example is simply the collection of singleton sets containing a natural number. So each set $A_i := \{i\} \forall i \in \mathbb{N}$. Since $\mathbb{N}$ is countably infinite, so the collection of sets. Each set is definitely finite, because they all contain just one element. Finally, the union of the collection of sets is equal to $\mathbb{N}$, which is not a finite set.

Exercise 6

a) Compute $f(4/15)$. Find q such that $f(q) = 108$.
b) Use the **Theorem** to prove that f is a bijection.

See the assignment PDF for the full assignment specification and theorem.

a) $\frac{4}{15}$, if written as a product of prime factors, is equal to $\frac{2^2}{3^1 \cdot 5^1}$. Since this fraction is not a natural number, we have to use the second part of the definition of f. So, $f(q) = 2^{2 \cdot 2} \cdot 3^{2 \cdot 1 - 1} \cdot 5^{2 \cdot 1 - 1} = 240$.

For the inverse of f, it is still necessary to compute the factorization in prime numbers. Using the powers of the primes we can deduce whether the prime present is, if applicable, part of either the numerator or the denominator. $180 = 2^2 \cdot 3^2 \cdot 5^1$. Because of the way f is defined, we know that all the prime factors with an even power are part of the numerator and all prime factors with an odd power are part of the denominator (except 1, which just maps to itself). When we backtrack using this information, we then get the following fraction: $\frac{2^1 \cdot 3^1}{5^1} = \frac{6}{5}$.

b) In order to prove that f is a bijection, we have to prove that f is injective and surjective.

Injectivity: We want to show that f is 1-1, i.e. $f(x_1) = f(x_2) \implies x_1 = x_2$.

So, let's assume that for any $x_1, x_2 \in \{q > 0 : q \in \mathbb{Q}\}$, $f(x_1) = f(x_2)$. Since the function f has 3 parts, based on the input, we have to prove this statement for those 3 parts separately as well. First, the easiest case, where the input set is $\{1\}$. Then, $f(x) = 1 \forall x$, so f is injective.

For the case where $x \in \mathbb{N} \backslash \{1\}$, $f(x) := p_1^{2r_1} \cdots p_N^{2r_N}$. We know from the **Theorem** that any fraction can be uniquely written as a product of prime factors with exponents, so when we assume $f(x_1) = f(x_2)$, we can also assume that x_1 and x_2 have a unique prime factorization associated with them. So let's assume that $f(x_1) = f(x_2)$. This means that $p_1^{2r_1} \cdots p_N^{2r_N} = q_1^{2s_1} \cdots q_M^{2s_M}$, where $p_i^{r_i}$ and $q_j^{s_j}$ denote the prime factors for both sides. We can further expand this expression into:

$$p_1^{r_1} \cdot p_1^{r_1} \cdots p_N^{r_N} \cdot p_N^{r_N} = q_1^{s_1} \cdot q_1^{s_1} \cdots q_M^{s_M} \cdot q_M^{s_M} \implies \tag{6.1}$$

$$p_1^{r_1} \cdots p_N^{r_N} \cdot p_1^{r_1} \cdots p_N^{r_N} = q_1^{s_1} \cdots q_M^{s_1} \cdot q_1^{s_1} \cdots q_N^{s_M} \implies \tag{6.2}$$

$$x_1 \cdot x_1 = x_2 \cdot x_2 \tag{6.3}$$

Because we know that each fraction constitutes a unique prime factorization, we also know that x_1 and x_2 are uniquely derived. This is why the implications in the equation above hold. Because both x_1 and $x_2 > 0$, $x_1 = x_2$.

Now for the case where $x \in \mathbb{Q}\backslash\mathbb{N}$. Then $f(x) := p_1^{2r_1} \cdots p_N^{2r_N} q_1^{2s_1-1} \cdots q_M^{2s_M-1}$, using the unique factorization derived from the **Theorem**. So again, we assume that for any $x_1, x_2 \in \mathbb{Q}\backslash\mathbb{N}$, $f(x_1) = f(x_2)$. Using the definition of f, we get: $p_1^{2r_1} \cdots p_N^{2r_N} q_1^{2s_1-1} \cdots q_M^{2s_M-1} = v_1^{2t_1} \cdots v_n^{2t_n} w_1^{2u_1-1} \cdots w_m^{2u_m-1}$.[1] Expanding this expression further, we get:

$$\frac{p_1^{2r_1}}{p_1} \cdots \frac{p_N^{2r_N}}{p_N} \frac{q_1^{2s_1}}{q_1} \cdots \frac{q_M^{2s_M}}{q_M} = \frac{v_1^{2t_1}}{v_1} \cdots \frac{v_n^{2t_n}}{v_n} \frac{w_1^{2u_1}}{w_1} \cdots \frac{w_m^{2u_m}}{w_m} \implies \tag{6.4}$$

$$\frac{p_1^{r_1} \cdot p_1^{r_1}}{p_1} \cdots \frac{p_N^{r_N} \cdot p_N^{r_N}}{p_N} \frac{q_1^{s_1} \cdot q_1^{s_1}}{q_1} \cdots \frac{q_M^{s_M} \cdot q_M^{s_M}}{q_M} = \frac{v_1^{t_1} \cdot v_1^{t_1}}{v_1} \cdots \frac{v_n^{t_n} \cdot v_n^{t_n}}{v_n} \frac{w_1^{u_1} \cdot w_1^{u_1}}{w_1} \cdots \frac{w_m^{u_m} \cdot w_m^{u_m}}{w_m} \implies \tag{6.5}$$

$$\frac{x_1 \cdot x_1}{p_1 \cdots p_N \cdot q_1 \cdots q_M} = \frac{x_2 \cdot x_2}{v_1 \cdots v_n \cdot w_1 \cdots w_m} \tag{6.6}$$

I'm kinda stuck at this point. I see that this is definitely injective, since the way the exponents are defined, you will always know which prime factors belong to the numerator or to the denominator. But I fail to prove this using the direct definition of f like we could do for the natural numbers. This is because the products of the denominators in the last equation are not unique. So maybe I simplified them too much and shouldn't try and write them in terms of x_1 and x_2 like we did earlier, and try and focus more on just the exponents, but I feel it becomes really hard to show that $x_1 = x_2$ that way.

Surjectivity: We want to show that f is onto, i.e. $f(\{q > 0 : q \in \mathbb{Q}\}) = \mathbb{N}$.

In order to prove this, we will take an arbitrary $y \in \mathbb{N}$, and show that $\exists x : f(x) = y$. We know from the **Theorem** that y can be written as a product of unique prime factors, $p_1^{r_1} \cdots p_N^{r_N}$. From the definition of f we know that if the exponents of the prime factors r are even, they belong to the numerator of x and if the exponents are odd, they belong to the denominator of x. If there are no prime factors with odd exponents, x will be a natural number. If $y = 1$, $x = 1$.

We will now only consider the case that y is a prime factorization with factors with odd exponents [2]. Then, we can find x in the following way: we multiply each prime factor

[1] The super- and subscripts become a bit abracadabra, but I think everything is unique and readable this way.

[2] The case for a prime factorization with solely even exponents can be backtracked in a similar fashion, just without the case for odd exponents and making x a fraction.

$p_i^{2r_i-1}$ with p_i and take the square root. We know that the square root of $p_i^{2r_i}$ is defined, since the exponent is multiplied by a factor 2, which the root negates. This will yield a prime factorization that we will put in the denominator of a fraction. We do the same for the prime factors with even exponent, but without multiplying with p_i. The prime factors we gain like that we put as a product in the numerator of the fraction. So, we gain a fraction with both the numerator and the denominator consisting of products of prime numbers, which are natural numbers, and so the fraction is positive and in fact a fraction. □

Part II

Assignment 2

Exercise 1.1.1

> Let F be an ordered field and $x, y, z \in F$. If $x < 0$ and $y < z$, then $xy > xz$.

So let's assume the premise. F is an ordered field and $x, y, z \in F$, and we choose x, y and z such that $x < 0$ and $y < z$.

From $x < 0$ it follows that $(-x) > 0$. From $y < z$ it follows that $0 < z - y$. From both of these, we can conclude that $0 < (-x)(z - y)$. Working out the right side with the distributive law, gives $0 < (-x \cdot z) - (-x \cdot y)$. Using $-1 \cdot -1 = 1$, gives $0 < (-xz) - (-xy)$, thus $0 < xy - xz$. The right part can be split again: $xz < xy$. Then, the < can be flipped, which gives $xy > xz$. $\square$

Exercise 1.1.2

> Let S be an ordered set. Let $A \subset S$ be a non-empty finite subset. Then A is bounded. Furthermore, $\inf A$ exists and in in A and $\sup A$ exists and is in A.

In order to prove that A is bounded, we have to prove that it has an upper and a lower bound. Let us prove that A is bounded above first.

In particular, we have to prove that $\exists a \in A$ such that $x \leq b$ for all $x \in E$. Since A is non-empty and finite, we can use induction on the cardinality of A, since that will always be some natural number n. So, we have to prove two cases: the base case, where $|A| = 1$, and the inductive step, where we will assume that when A has an upper bound when it has cardinality m, then it also has an upper bound when its cardinality is equal to $m + 1$.

The base case is quite simple; if $A = \{x\}$, then x is the greatest element and A has an upper bound. Now for the inductive step. We assume that for some set $B \subset S$ with cardinality m, B is bounded above. Thus, there is some $b \in B$ such that b is greater than all other elements in B. Now, let's add a new element $h \in S$ to B, such that h is distinct from all elements already in B and the cardinality of B is now $m + 1$. Then, since S is well ordered, we can compare h also to b. Either h is greater than this b, in which case h is the new greatest element, or it is less than b, in which case b stays the greatest element of B. In both cases however, B remains bounded above. $\square$

A similar argument can be made to prove the existence of the lower bound, the supremum of A in A and the infimum of A in A [3]. This will be left to the reader.

[3] It might even be that I have already proven that A has a supremum present in A. Then that's also good enough to show that A is bounded, since in order for A to have a supremum, it must also be bounded.

Exercise 1.1.5

> Let S be an ordered set. Let $A \subset S$ and suppose b is an upper bound for A. Suppose $b \in A$. Show that $b = \sup A$.

So, let S be an ordered set, with $A \subset S$ and $b \in A$ being an upper bound for A. Since b is an upper bound, $a \leq b$ for all $a \in A$. Since $b \in A$ as well, we know that there is some element in A which is the greatest element of them all, and all other elements are smaller.

Now let's assume that $b \neq \sup A$. Then either some other element of A is the supremum, which would imply that b is not larger than this element, which is a contradiction. The other possibility is that there is an element $c \in S \backslash A$ that is the supremum. Because S is ordered, c must either be greater than, smaller than or equal to b. If $c < b$, c is not an upper bound of A and thus also not its supremum. If $c > b$, then b is an upper bound that is smaller than c and therefore c cannot be the supremum. The only option left is that $c = b$, and therefore $b = \sup A$. □

Exercise 1.1.6

> Let S be an ordered set. Let $A \subset S$ be nonempty and bounded above. Suppose $\sup A$ exists and $\sup A \notin A$. Show that A contains a countably infinite subset.

Let S be an ordered set, with $A \subset S$ nonempty and bounded above. We assume that $b = \sup A$ exists and $b \notin A$ ($\implies b \in S \backslash A$). We are asked to show this then implies that $\exists X \subset A$ such that $|X| \geq |\mathbb{N}|$. We will prove this with a proof by contradiction.

We assume that no such set X exists, i.e. $|X| < |\mathbb{N}|$. So, A also doesn't have to countably infinite anymore. Since $b \notin A$ and A is ordered, finite and nonempty, there is a greatest element $a \in A$ such that $a < b$ and $x < a \ \forall x \in A$. So b is in fact not the least upper bound of A, which is in contradiction with our assumption earlier. Ergo, $|X| \geq |\mathbb{N}|$. Since X then is at least of the same cardinality as $\mathbb{N}$, it must also contain a countably infinite subset. □

Exercise 1.2.7

> Prove the arithmetic-geometric mean inequality. That is, for two positive real numbers x, y, we have
> $$\sqrt{xy} \leq \frac{x+y}{2}.$$
> Furthermore, equality occurs if and only if $x = y$.

Let us prove the first statement first. So we let $x, y \in \mathbb{R}$ such that $x, y > 0$. Then we will prove the statement by contradiction. Hence, we assume that

$$\sqrt{xy} > \frac{x+y}{2}.$$

We can multiply both sides with 2. This results in $2\sqrt{xy} > x + y$. We can pull the left part into the right, so we get $0 > x - 2\sqrt{xy} + y$. We can restructure the right side to $0 > (\sqrt{x} - \sqrt{y})^2$. We know that $0 \leq z^2, \forall z \in \mathbb{R}$, so this is a contradiction. $\square$

Now to prove the second statement. We assume $x = y$ is a positive real number. Then,

$$\sqrt{xy} = \sqrt{x^2} = x = \frac{2x}{2} = \frac{x+y}{2}. \quad \square$$

Exercise 1.2.9

> Let A and B be two nonempty bounded sets of real numbers. Define the set $C := \{a+b : a \in A, b \in B\}$. Show that C is a bounded set and that
>
> $$\sup C = \sup A + \sup B \text{ and}$$
> $$\inf C = \inf A + \inf B.$$

First, let us show that C is a bounded set. Since A and B are both subsets of $\mathbb{R}$, which is an ordered field, all elements of C must also be real numbers. Let a be an upper bound for A and b be an upper bound for B. So $x \leq a \; \forall x \in A$ and $y \leq b \; \forall y \in B$. Since C is defined as the sum of any element in A with any element in B, an upper bound of C, c, can be found as $c \leq a + b$. A similar argument can be made for the lower bound of C, which makes C bounded. $\square$

To prove that $\sup C = \sup A + \sup B$, we will show that $\sup C \geq \sup A + \sup B$ and $\sup C \leq \sup A + \sup B$.

Let $a = \sup A$ and $b = \sup B$. So $x \leq a$ for all $x \in A$ and $y \leq b$ for all $y \in B$. Then, $x + y \leq a + b$. Since $z \leq x + y$ for all $z \in C$ because of the definition of C, $z \leq a + b$. In other words, $\sup C \leq \sup A + \sup B$.

Now to prove the other direction. Let $c = \sup C$. So $z \leq c$ for all $c \in C$. Since all elements in C are the sum of an element $x \in A$ and $y \in B$, $x + y \leq c$ for all x, y. The least upper bound for these x and y can be given by the supremum; $x \leq \sup A$ and $y \leq \sup B$. So, $\sup A + \sup B \leq c \implies \sup A + \sup B \leq \sup C$, completing the equality. $\square$

A similar argument can be given for the infimum, which is left to the reader.

Exercise 7

> Let
> $$E = \{x \in \mathbb{R} : x > 0 \text{ and } x^3 < 2\}.$$
>
> *a)* Prove that E is bounded above.
> *b)* Let $r = \sup E$ (which exists by part a)). Prove that $r > 0$ and $r^3 = 2$.
> *Hint:* Adapt the proof used in Example 1.2.3.

So, let E and r be defined as in the exercise statement. Then:

a) $x \leq 2$ is an upper bound for E, as $2 \cdot 2 \cdot 2 = 8$. So, E is bounded above.

b) As $1 \in E$, $r \geq 1 > 0$, so the first part of the statement holds. In order to show that $r^3 = 2$, we want to show that $r^3 \leq 2$ and $r^3 \geq 2$ hold.

First, let's show that $r^3 \geq 2$. We will take a similar approach as in Example 1.2.3 from the textbook. So, take a positive number s such that $s^3 < 2$. We wish to find an $h > 0$ such that $(s+h)^3 < 2$. As $2 - s^3 > 0$, we have $\frac{2-s^3}{3s^2+3s+1} > 0$. Choose an $h \in \mathbb{R}$ such that $0 < h < \frac{2-s^3}{3s^2+3s+1}$. Furthermore, assume $h < 1$. Estimate,

$$\begin{aligned}
(s+h)^3 - s^3 &= h(3s^2 + 3sh + h^2) \\
&< h(3s^2 + 3s + 1) && \text{(since } h < 1\text{)} \\
&< 2 - s^3 && \left(\text{since } h < \frac{2-s^3}{3s^2+3s+1}\right).
\end{aligned}$$

Therefore, $(s+h)^3 < 2$. Hence $s + h \in E$, but as $h > 0$, we have $s + h > s$. So $s < r = \sup E$. As s was an arbitrary positive number such that $s^3 < 2$, it follows that $r^3 \geq 2$.

Now take an arbitrary positive number s such that $s^3 > 2$. We wish to find an $h > 0$ such that $(s-h)^3 > 2$ and $s - h$ is still positive. As $s^3 - 2 > 0$, we have that $\frac{s^3-2}{3s^2+1} > 0$. Let $h := \frac{s^3-2}{3s^2+1}$, and check that $s - h = s - \frac{s^3-2}{3s^2+1} = \frac{2s^3+s+2}{3s^2+1} > 0$. Assume that $h < 1$. Estimate,

$$\begin{aligned}
s^3 - (s-h)^3 &= h(3s^2 - 3sh + h^2) \\
&< h(3s^2 + h^2) && \text{(since } s > 0 \text{ and } h > 0\text{)} \\
&< h(3s^2 + 1) && \text{(since } h < 1\text{)} \\
&= s^3 - 2 && \text{(because of the definition of } h\text{)}.
\end{aligned}$$

By subtracting s^3 from both sides and multiplying by -1, we find $(s-h)^3 > 2$. Therefore, $s - h \notin E$. Moreover, if $x \geq s - h$, then $x^3 \geq (s-h)^3 > 2$ (as $x > 0$ and $s - h > 0$) and so $x \notin E$. Thus, $s - h$ is an upper bound for E. However, $s - h < s$, or in other words, $s > r = \sup E$. Hence, $r^3 \leq 2$.

Together, $r^3 \geq 2$ and $r^3 \leq 2$ imply $r^3 = 3$. □

Part III

Assignment 3

Exercise 1

Suppose $x, y \in \mathbb{R}$ and $x < y$. Prove that there exists $i \in \mathbb{R} \backslash \mathbb{Q}$ such that $x < i < y$.

If either x or y (or both) are not rational numbers, we can simply take the average like so: $\frac{x+y}{2}$, in a similar way we did for the rationals. Since x or y isn't rational, the resulting fraction will also not be a rational and this proves the statement.

Now if $x, y \in \mathbb{Q}$, we cannot use this average trick, because the resulting fraction will be a rational itself and so it doesn't satisfy the restriction that it must be in $\mathbb{R} \backslash \mathbb{Q}$. So we have to take a different approach.

Let $x, y \in \mathbb{Q}$ with $x < y$ and $m := \frac{x+y}{2}$, so $x < m < y$. Then, let $X = \{a \in \mathbb{R} : x < a < m\}$ and let $Y = \{b \in \mathbb{R} : m < b < y\}$. Since $x < m$ and $m < y$, these are nonempty and they are bounded, because of the restrictions $x < a < m$ and $m < b < y$. So, there exists $k \in X$, that is not rational such that $x < k < m$ and there exists $h \in Y$ that is not rational such that $m < h < y$. Pick either k or h as i, since $x < k < m < h < y$. □

Exercise 2

Let $E \subset (0, 1)$ be the set of all real numbers with decimal representation using only the digits 1 and 2:

$$E := \{x \in (0, 1) : \forall j \in \mathbb{N}, \exists d_j \in \{1, 2\} \text{ such that } x = 0.d_1 d_2 ...\}$$

Prove that $|E| = |\mathcal{P}(\mathbb{N})|$.

As a hint to this exercise: *Consider the function* $f : E \to \mathcal{P}(\mathbb{N})$ *such that if* $x \in E, x = 0.d_1 d_2 ...$,

$$f(x) = \{j \in \mathbb{N} : d_j = 2\}.$$

In order to prove that 2 sets are of equal cardinality, we need to prove that there is a bijective function between the 2 sets. In this case, the aforementioned hint function does the trick. Non-formally speaking, it is exactly what we are looking for: it is a (weird) representation of the power set of natural numbers, in that for every decimal, represented by a natural number, it is decided if that decimal is a 2 or a 1. This is similar to the actual power set of the natural numbers, in which for every natural number it is decided whether the number is in a subset or not.

Now for a formal proof. To show that f is bijective, we need to show that it is surjective and injective.

Injectivity of f In order to show that f is injective, we have to show that for every $x \in E$, there is a unique $y \in \mathcal{P}(\mathbb{N})$ for the function, by showing that $f(a) = f(b) \implies a = b$.

So, let's assume that for some $a, b \in E$, $f(a) = f(b)$. So, there two sets of natural numbers $\{a_1, a_2, ..., a_n\} = \{b_1, b_2, ..., b_m\}$. Equality in sets means that every element that is present in the one set, is present in the other, and vice versa. No element that is present in either set, is missing in the other. So, in this case, both sets will represent the same sequence of digits that are 2. Because the only other option for digits is 1, that means the complete digital representation of a and b are known, unique and the same. This concludes the proof for injectivity.

Surjectivity of f To prove surjectivity, we need to prove that for any arbitrary $y \in \mathcal{P}(\mathbb{N})$, there exists a corresponding $x \in E$ such that $f(x) = y$.

So, take an arbitrary $y = \{y_1, y_2, ..., y_n\}$, where each $y_i \in \mathbb{N}$ and thus $y \in \mathcal{P}(\mathbb{N})$. Then, corresponding $x \in E$ can be constructed easily as follows. Take a decimal number $0.d_1d_2...$ and turn every decimal d_i for which $i \in y$ into a 2, and every other decimal into a 1. Since every decimal can only be a 1 or 2, this handles every decimal correctly. Also, $f(x)$ will be in $\mathcal{P}(\mathbb{N})$.

Since, f is 1-to-1 and onto, f is bijective. Then, because there exists a bijective function from E to $\mathcal{P}(\mathbb{N})$, $|E| = |\mathcal{P}(\mathbb{N})|$. □

Exercise 3

(a) Let A and B be two disjoint, countably infinite sets. Prove that $A \cup B$ is countably infinite.
(b) Prove that the set of irrational numbers, $\mathbb{R} \backslash \mathbb{Q}$, is uncountable. You may use the facts discussed in the lectures that $\mathbb{R} \backslash \mathbb{Q}$ is infinite and $\mathbb{R}$ is uncountable without proof.

(a) So let A and B be two disjoint, countably infinite sets. Since these sets are countably infinite, a bijective function to $\mathbb{N}$ exists for both functions separately. It is then straightforward to map both these function together to $\mathbb{Z}$ instead, in the following way. Let f be the bijective function such that $f : A \to \mathbb{N}$ and let g be the bijective function such that $g : B \to \mathbb{N}$. Then, we can define a new function $h : A \cup B \to \mathbb{Z}$ as

$$\begin{aligned} h(x) &= f(x) \text{ if } x \in A \\ &= -g(x) \text{ if } x \in B. \end{aligned}$$

Since $A \cap B = \emptyset$, this function is unambiguously defined. Since $\mathbb{Z}$ is countably infinite, $A \cup B$ is countably infinite as well. □

(b) Because of part (a), we know that if we have two disjoint, countably infinite sets and join them, the result is still countably infinite. The opposite must then also be true: if we have a countably infinite set and we divide it into two disjoint subsets, both of which are infinite, then they still must be countable.

So then, for $\mathbb{R} \backslash \mathbb{Q}$, we know that $\mathbb{R}$ is uncountably infinite. So when we split it into rational and irrational subsets, from which we know that $\mathbb{Q}$ is countably infinite, $\mathbb{R} \backslash \mathbb{Q}$ must be at least and at most uncountably infinite. □

Exercise 4

Let A be a subset of $\mathbb{R}$ which is bounded above, and let a_0 be an upper bound for A. Prove that $a_0 = \sup A$ if and only if for every $\varepsilon > 0$, there exists $a \in A$ such that $a_0 - \varepsilon < a$.

Let $A \subset \mathbb{R}$, with A bounded above by a_0. So, we have to prove the implication both ways. First, let's prove that the implication to the right ($\rightarrow$).

Assume that $a_0 = \sup A$, so for all $a \in A$, $a \leq a_0$. Also, let $\varepsilon > 0$. If $a_0 \in A$, then we pick a_0 as a and get $a_0 - \varepsilon < a_0$, which holds $\forall \varepsilon > 0$. If $a_0 \notin A$, then we choose a as the average of a_0 and $a_0 - \varepsilon$, which is definitely smaller than a_0. We are allowed to pick this as a, because we assume without loss of generality that $a \geq \inf A$. Then we get

$$\begin{aligned} a_0 - \varepsilon &< \frac{a_0 + a_0 - \varepsilon}{2} \\ &< a_0 - \frac{\varepsilon}{2} \implies \\ -\varepsilon &< -\frac{\varepsilon}{2}. \end{aligned}$$

Since $\varepsilon > 0$, this always holds.

Now for the implication to the left ($\leftarrow$).

Assume now that the right side is true, i.e. let's assume that $\forall \varepsilon > 0$, there exists $a \in A$ such that $a_0 - \varepsilon < a$. Again, let us first investigate the case where $a_0 \in A$. Well certainly still, if a_0 is an upper bound for A and it is also part of the set itself, it must be the supremum[4].

Then, let's assume that $a_0 \notin A$. Now, for all positive ε, we know there exists an $a \in A$ such that $a \neq a_0$ and $a_0 - \varepsilon < a$. Let us assume then that this implies that $a_0 \neq \sup A$ and try to come to a contradiction. So, then there must be some $b = \sup A$, which has as consequence that $a < b < a_0$, since b is still an uppoer bound of A (and $a_0 \notin A$). Then, since $b > a$, we can pick $a = b - \varepsilon < b$. So, from our initial assumption we get $b - \varepsilon < a_0 - \varepsilon < b - \varepsilon \implies b < a_0 < b$, which is a false statement. So, $a_0 = \sup A$.

Since the implication holds both ways, the equivalence is proven. $\square$

[4]Proven in earlier exercise.

Exercise 5

(a) Let $a, b \in \mathbb{R}$ with $a < b$. Prove that the sets $(-\infty, a), (a, b)$ and (b, ∞) are open.

(b) Let Λ be a set (not necessarily a subset of $\mathbb{R}$), and for each $\lambda \in \Lambda$, let $U_\lambda \subset \mathbb{R}$. Prove that if U_λ is open for all $\lambda \in \Lambda$ then the set

$$\bigcup_{\lambda \in \Lambda} U_\lambda = \{x \in \mathbb{R} : \exists \lambda \in \Lambda \text{ such that } x \in U_\lambda\}$$

is open.

(c) Let $n \in \mathbb{N}$, and let $U_1, \ldots, U_n \subset \mathbb{R}$. Prove that if $U_1, \ldots, U_n$ are open then the set

$$\bigcap_{m=1}^{n} U_m = \{x \in \mathbb{R} : x \in U_m \text{ for all } m = 1, \ldots, n\}$$

is open.

(d) Is the set of rationals $\mathbb{Q} \subset \mathbb{R}$ open? Provide a proof to substantiate your claim.

(a) Since $\mathbb{R}$ is open, it is clear that $(-\infty, a)$ and (b, ∞) are open to the left and right respectively as well. Also, their respective right and left side are present in (a, b) as well, so we will only prove it for this case. The other cases follow logically.

Let $a, b \in \mathbb{R}$ such that $a < b$. We want to show that for all $x \in (a, b)$ there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset (a, b)$. Since for all $y \in (a, b)$ such that $x - \varepsilon < y < x + \varepsilon$ this statement will recursively hold, we only need to prove that there exists $\varepsilon > 0$ such that $a < x - \varepsilon$ and $x + \varepsilon < b$. Then we can pick a fitting ε in the following way, depending if x is closer to a or to b, formalized as follows.

If $x - a = b - x \implies 2x = b + a \implies x = \frac{b+a}{2}$, then x is precisely between a and b and we can pick ε to be $\frac{b-a}{4}$. Then $x + \varepsilon < b$ and $x - \varepsilon > a$.

When $x - a < b - x$, then x will be closer to a then to be and ε is bounded more by x's proximity to a than to b, i.e. $\varepsilon < x - a$. So we can pick $\varepsilon = \frac{x-a}{2} < x - a$. Then $x - \varepsilon = x - \frac{x-a}{2} = \frac{x+a}{2}$. Since $x > a$, $\frac{x+a}{2} > a$. For the other side, $x + \varepsilon = x + \frac{x-a}{2} < x + \frac{b-x}{2} = \frac{x+b}{2} < b$ since $x < b$. So for both sides, we have shown that there exists an ε such that both $x - \varepsilon, x + \varepsilon \in (a, b)$. All elements inbetween $x - \varepsilon$ and $x + \varepsilon$ will also definitely be in (a, b).

The argument when $b - x < x - a$ is very similar and will be left to the reader. Then, for all $x \in (a, b)$, the statement is proven. □

(b) Non-formally speaking, in this exercise we want to prove that any union of open sets in $\mathbb{R}$ is open itself. In order to make this formal, we will assume that U_λ is open for all $\lambda \in \Lambda$ and follow the definition as presented.

So, let us assume that U_λ is open for all $\lambda \in \Lambda$. This means that for every $x_\lambda \in U_\lambda$, there exists an $\varepsilon > 0$ such that $(x_\lambda - \varepsilon, x_\lambda + \varepsilon) \subset U_\lambda$. To prove that $\bigcup_{\lambda \in \Lambda} U_\lambda$ is open, we need to show that the same property holds for all y in this set. But since the union between some sets is defined as the set that holds all the elements that any of these sets hold, this is trivial: for any $y \in \bigcup_{\lambda \in \Lambda}$ for which we want to know what ε we need to show that the union is open around that y, we just pick the corresponding ε for the subset U_λ which was open. Since all elements in that U_λ are also in the union, this must certainly be the case. □

(c) Similarly to the previous exercise, non-formally speaking we want to prove that any intersection of open sets in $\mathbb{R}$ is open itself. This is not as trivial as in the previeous exercise however: since every set that is added as an intersection poses another restriction, we don't have the immediate guarantee that every ε from the subsets will also be a well-defined element for the intersection set.

Now formally. Let $n \in \mathbb{N}$ and let $U_1, ..., U_n \subset \mathbb{R}$. We assume that all $U_1, ..., U_n$ are open. Then we will prove that $\bigcap_{m=1}^{n} U_m$ is open by induction over n.

For the base case, let $n = 1$. Then the intersection set is equivalent to U_1. Since U_1 is open, then so is the intersection set.

For the inductive step, we assume that the intersection set is open for a certain $n = h$, i.e. there exists an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon)$ is open for every $x \in \bigcap_{m=1}^{h} U_m$. Now, we will add one additional open set to this intersection, U_{h+1}, such that n become $h + 1$. Note that $h + 1 \in \mathbb{N}$. Let the new intersection set be denoted as $\bigcap'$, and the old one as $\bigcap$. Then in order to find an $\varepsilon > 0$ for every $x \in \bigcap'$ such that $(x - \varepsilon, x + \varepsilon)$, we take the smallest of ε's for that x compared between $\bigcap$ and U_{h+1}. Since $x \in \bigcap'$, we know that $x \in \bigcap$ and $x \in U_{h+1}$. Then the smallest accomponying ε always gives a well-defined open set inside of $\bigcap'$ because $|(x + \varepsilon_1) - (x - \varepsilon_1)| < |(x + \varepsilon_2) - (x - \varepsilon_2)|$ if $\varepsilon_1 < \varepsilon_2$, and thus $\bigcap'$ is open itself. □

(d) No, $\mathbb{Q}$ is not open in $\mathbb{R}$. This is because we can't find an $\varepsilon > 0$ such that for every $q \in \mathbb{Q}$, $(q - \varepsilon, q + \varepsilon) \subset \mathbb{Q}$. We know that $\mathbb{Q}$ is dense in $\mathbb{R}$, but as we have proven in exercise 1, the converse is also true. For every real numbers, we can find a real number inbetween that is not a rational number. So, we cannot pick an $\varepsilon > 0$ such that there is an interval around x that itself is completely contained in $\mathbb{Q}$. For every ε we pick, we can always find a real number r such that $x < r < x + \varepsilon$ and $x - \varepsilon < r < x$. So, $\mathbb{Q}$ is not open. □

Exercise 6

Prove that

$$\lim_{n\to\infty} \frac{1}{20n^2 + 20n + 2020} = 0.$$

In order to prove that this limit holds, we need to show that a function $\{x_n\}$ converges to x, i.e. if for all $\varepsilon > 0$, $\exists M \in \mathbb{N}$ such that $\forall n \geq M$ the following inequality holds: $|x_n = x| < \varepsilon$.

Let $\varepsilon > 0$. We choose $M \in \mathbb{N}$ such that $\frac{1}{M} < \varepsilon$ (Archimedean Property). Then for all $n \geq M$, $|\frac{1}{20n^2+20n+2020} - 0| = \frac{1}{20n^2+20n+2020} \leq \frac{1}{n^2+n} \leq \frac{1}{n} \leq \frac{1}{M} < \varepsilon$. □